PRODUCT OF FUNCTIONS IN BMO AND $\mathcal{H}^1$ IN NON-HOMOGENEOUS SPACES

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Abstract

Under the assumption that the underlying measure is a non-negative Radon measure which only satisfies some growth condition and may not be doubling, we define the product of functions in the regular BMO and the atomic block $\mathcal{H}^1$ in the sense of distribution, and show that this product may be split into two parts, one in L^1 and the other in some Hardy-Orlicz space.

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1 Introduction

In their paper [1], Bonami, Iwaniec, Jones and Zinsmeister defined the product of functions $f \in BMO(\mathbb{R}^n)$ and $h \in \mathcal{H}^1(\mathbb{R}^n)$ as a distribution operating on a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by the rule

$$\langle f \times h, \varphi \rangle := \langle f\varphi, h \rangle . \tag{1.1}$$

They proved that such distribution can be written as the sum of a function in $L^1(\mathbb{R}^n)$ and a distribution in a Hardy-Orlicz space $\mathcal{H}^{\wp}(\mathbb{R}^n, \nu)$ where

$$\wp(t) = \frac{t}{\log(e+t)} \text{ and } d\nu(x) = \frac{dx}{\log(e+|x|)}. \tag{1.2}$$

Bonami and Feuto in [2] considered the case where $BMO(\mathbb{R}^n)$ is replaced by its local version $\mathfrak{bmo}(\mathbb{R}^n)$ introduced by Golberg in [3], and proved that in this case, the weighted Hardy-Orlicz space is replaced by a space of amalgam type in the sense of Wiener [4]. Following the idea in [1] and [2], the author in [5] generalized this result in the setting of space of homogeneous type $(\mathcal{X}, d, \mu)$. We recall that a space of homogeneous type is a non-empty set $\mathcal{X}$ equipped with a quasi metric d and a positive Radon measure μ such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)),\ x \in \mathcal{X},\ r > 0 \tag{1.3}$$

where $B(x,r) = \{y \in \mathcal{X} : d(x,y) < r\}$ is the ball centered at x and having radius r.

This doubling condition is an essential assumption for most results in classical function spaces, Calderón-Zygmund theory and operators theory. However, it has been shown recently (see [6], [7], [8], [9] and [10], and the reference therein) that one can drop the doubling condition and still obtain interesting results in the classical Calderón-Zygmund theory and on the classical Hardy and BMO spaces. In particular, Tolsa in [7] introduced, when the measure satisfies only the growth condition (1.4), the regular bounded mean oscillation space $RBMO(\mu)$ and its predual space $\mathcal{H}^{1,\infty}_{atb}(\mu)$. He showed that these spaces have similar properties to those of the classical BMO and $\mathcal{H}^1$ defined for doubling measures.

The purpose of this paper is to define the product of function in $RBMO(\mu)$ and $\mathcal{H}^{1,\infty}_{atb}(\mu)$ in the sense of distribution, and to prove that some results obtained in [2], [5] and [1] are valid in this context. To make our idea clear, let us give some notations and definitions.

Let n, d be some fixed integers with $0 < n \leq d$. We consider $(\mathbb{R}^d, |\cdot|, \mu)$, where $|\cdot|$ is the Euclidean metric and μ a positive Radon measure that only satisfies the following growth condition

$$\mu(B(x,r)) \leq C_0 r^n, \text{ for all } x \in \mathbb{R}^d \text{ and } r > 0, \tag{1.4}$$

where $C_0 > 0$ is an absolute constant. Throughout the paper, by a cube $Q \subset \mathbb{R}^d$, we mean a closed cube with sides parallel to the axis and centered at some point x_Q of $\mathrm{supp}(\mu)$, and if $\|\mu\| < \infty$, we allow $Q = \mathbb{R}^d$ too.

If Q is a cube, we denote by $\ell(Q)$ the side length of Q and for $\alpha > 0$, we denote αQ the cube with same center as Q, but side length α times as long. We will always choose the constant C_0 in (1.4) such that for all cubes Q, we have $\mu(Q) \leq C_0 \ell(Q)^n$.

For two fixed cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$S_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{\ell^n(2^k Q)} \tag{1.5}$$

where $N_{Q,R}$ is the smallest positive integer k such that $\ell(2^k Q) \geq \ell(R)$ (in the case $R = \mathbb{R}^d \neq Q$, we set $N_{Q,R} = \infty$).

For a fixed $\rho > 1$ and $p \in (1,\infty]$, a function $b \in L^1_{loc}(\mu)$ is called a p-atomic block if

(i) there exists some cube R such that supp $b \subset R$,

(ii) $\int_{\mathbb{R}^d} b \, d\mu = 0$,

(iii) there are functions a_j supported on cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \sum_{j=1}^{\infty} \lambda_j a_j$ and

$$\left\|a_j\right\|_{L^p(\mu)} \leq (\mu(\rho Q_j))^{\frac{1}{p}-1} \left(S_{Q_j,R}\right)^{-1}, \tag{1.6}$$

where we used the natural convention that $\frac{1}{\infty} = 0$. We put

$$|b|_{\mathcal{H}^{1,p}_{atb}(\mu)} := \sum_j \left|\lambda_j\right|. \tag{1.7}$$

Definition 1.1. *([7]) We say that $h \in \mathcal{H}^{1,p}_{atb}(\mu)$ if there are p-atomic blocks b_j such that*

$$h = \sum_{j=1}^{\infty} b_j \text{ with } \sum_{j=1}^{\infty} \left|b_j\right|_{\mathcal{H}^{1,p}_{atb}(\mu)} < \infty, \tag{1.8}$$

The atomic block Hardy space $\mathcal{H}^{1,p}_{atb}(\mu)$ is a Banach space when equipped with the norm $\|\cdot\|_{\mathcal{H}^{1,p}_{atb}(\mu)}$ defined by

$$\|h\|_{\mathcal{H}^{1,p}_{atb}(\mu)} = \inf \sum_{j=1}^{\infty} \left|b_j\right|_{\mathcal{H}^{1,p}_{atb}(\mu)},\ h \in \mathcal{H}^{1,p}_{atb}(\mu), \tag{1.9}$$

where the infimum is taken over all possible decomposition of h into atomic blocks.

As it is proved in Proposition 5.1 and in Theorem 5.5 of [7], the definition of $\mathcal{H}^{1,p}_{atb}(\mu)$ does not depend on ρ and we have that, for all $1 < p < \infty$, the spaces $\mathcal{H}^{1,p}_{atb}(\mu)$ are topologically equivalent to $\mathcal{H}^{1,\infty}_{atb}(\mu)$. So in the sequel, we shall use the notation $\mathcal{H}^1(\mu)$ instead of $\mathcal{H}^{1,\infty}_{atb}(\mu)$, and take $\rho = 2$.

When $b \in L^1_{loc}(\mu)$ satisfies only Condition (i) and (iii) of the definition of atomic blocks, we say that it is a p-block and put $|b|_{\mathfrak{h}^1_{atb}(\mu)} = \sum_j \left|\lambda_j\right|$. Moreover, we say that h belongs to the local Hardy space $\mathfrak{h}^{1,p}_{atb}(\mu)$ (see [9]), if there are p-atomic blocks or p-blocks b_j such that

$$h = \sum_{j=1}^{\infty} b_j, \tag{1.10}$$

where $\sum_{j=1}^{\infty} \left|b_j\right|_{\mathfrak{h}^1_{atb}(\mu)} < \infty$, b_j is an atomic block if supp$b_j \subset R_j$ and $\ell(R_j) \leq 1$, and b_j is a block if supp$b_j \subset R_j$ and $\ell(R_j) > 1$. We define the $\mathfrak{h}^1_{atb}(\mu)$ norm of h by

$$\|h\|_{\mathfrak{h}^1_{atb}(\mu)} = \inf \sum_{j=1}^{\infty} \left|b_j\right|_{\mathfrak{h}^1_{atb}(\mu)}, \tag{1.11}$$

where the infimum is taken over all possible decompositions of h into atomic blocks or blocks.

The definition of local Hardy space is independent of $\rho > 0$ and for $1 < p < \infty$, we have $\mathfrak{h}^{1,p}_{atb}(\mu) = \mathfrak{h}^{1,\infty}_{atb}(\mu)$ (see Proposition 3.4 and Theorem 3.8 of [9]). This allow us to just denote it by $\mathfrak{h}^1(\mu)$ and consider also $\rho = 2$.

In Theorem 5.5 of [7] and Theorem 3.8 of [9], it is proved that the dual space of $\mathcal{H}^1(\mu)$ and $\mathfrak{h}^1(\mu)$ are respectively $RBMO(\mu)$ and its local version $\mathfrak{rbmo}(\mu)$ (see Section 2 for more explanations about these spaces).

Let $h = \sum_j b_j$ belonging to $\mathcal{H}^1(\mu)$, where the atomic block b_j is supported in the cube R_j and satisfies $b_j = \sum_i \lambda_{ij} a_{ij}$ for a_{ij}'s and λ_{ij}'s as in the definition of atomic blocks. For $f \in RBMO(\mu)$, we denote by $f_{\tilde{R}}$ the mean value of f over the cube $\tilde{R}$, which is an appropriate dilation of the cube R (see Section 2 for more explanation). We can see from the proof of Theorem 1.2 that the double series

$$\sum_{j=1}^{\infty} \left(f - f_{\tilde{R}_j}\right) b_j = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \lambda_{ij} \left(f - f_{\tilde{R}_j}\right) a_{ij}\right) \tag{1.12}$$

converges normally in $L^1(\mu)$, while

$$\sum_{j=1}^{\infty} f_{\tilde{R}_j} b_j = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} f_{\tilde{R}_j} \lambda_{ij} a_{ij}\right) \tag{1.13}$$

converges in the Hardy-Orlicz space $\mathcal{H}^{\wp}(\nu)$, where $\wp(t) = \frac{t}{\log(e+t)}$ and $d\nu(x) = \frac{d\mu(x)}{\log(e+|x|)}$. Since both convergence implies convergence in the sense of distribution, we define the product of f and h as the sum of both series by

$$f \times h = \sum_{j=1}^{\infty} \left(f - f_{\tilde{R}_j}\right) b_j + \sum_{j=1}^{\infty} f_{\tilde{R}_j} b_j. \tag{1.14}$$

It follows that

Theorem 1.2. *For f in $RBMO(\mu)$ and h in $\mathcal{H}^1(\mu)$, the product $f \times h$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion*

$$f \times h \in L^1(\mu) + \mathcal{H}^{\wp}(\nu). \tag{1.15}$$

When we replaced $RBMO(\mu)$ by its local version $\mathfrak{rbmo}(\mu)$ as define in [9] (see also [11]) we obtain the analogous of the result in [2]. We also obtain interesting results by replacing both $RBMO(\mu)$ and $\mathcal{H}^1(\mu)$ with their local version.

The paper is organized as follows, in Section 2 we recall the definition of the space $RBMO(\mu)$, its local version and some properties involved.

Section 3 is devoted to auxiliary results and prerequisites in Orlicz spaces while in Section 4 we give the proof of the main results and their extensions.

Throughout the paper, the letter C is used for non-negative constants that may change from one occurrence to another. Constants with subscript, such as C_0, do not change in different occurrences. The notation $A \approx B$ stands for $C^{-1}A \leq B \leq CA$, C being a constant not depending on the main parameters involved.

2 Prerequisite about $RBMO(\mu)$, $\mathfrak{rbmo}(\mu)$, $\mathcal{H}^1(\mu)$ and $\mathfrak{h}^1(\mu)$ spaces

Definition 2.1. *Let $\alpha > 1$ and $\beta > \alpha^n$, we say that a cube Q is an (α,β)-doubling cube if $\mu(\alpha Q) \leq \beta\mu(Q)$.*

It is proved in [7] that there are a lot of "big " doubling cubes and also a lot of "small" doubling cubes, this due to the facts that μ satisfies the growth Condition (1.4) and $\beta > \alpha^n$. More precisely, given any point $x \in$supp(μ) and $c > 0$, there exists some (α,β)-doubling cube Q centered at x with $\ell(Q) \geq c$.

On the other hand, if $\beta > \alpha^n$ then, for μ-a.e. $x \in \mathbb{R}^d$, there exists a sequence of (α,β)-doubling cubes $\{Q_k\}_{k\in\mathbb{N}}$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$.

In the following, for any $\alpha > 1$, we denote by β_α one of these big constants β. For definiteness, one can assume that β_α is twice the infimum of these β's.

Given $\rho > 1$, we let N be the smallest non-negative integer such that $2^N Q$ is (ρ,β_ρ)-doubling and we denote this cube by $\tilde{Q}$.

Definition 2.2. *([9]) Let $\rho > 1$ be some fixed constant.*

(a) Let $1 < \eta < \infty$. We say that $f \in L^1_{loc}(\mu)$ is in $RBMO(\mu)$ if there exists a non-negative constant C_2 such that for any cube Q,

$$\frac{1}{\mu(\eta Q)}\int_Q \left| f(x) - f_{\tilde{Q}}\right| d\mu(x) \leq C_2, \tag{2.1}$$

and for any two (ρ,β_ρ)-doubling cubes $Q \subset R$

$$|f_Q - f_R| \leq C_2 S_{Q,R}. \tag{2.2}$$

Let us put

$$\|f\|_{RBMO(\mu)} = \inf\{C_2 : (2.1) \text{ and } (2.2) \text{ hold}\}. \tag{2.3}$$

(b) Let $1 < \eta \leq \rho < \infty$. We say that $f \in L^1_{loc}(\mu)$ belongs to $\mathfrak{rbmo}(\mu)$ if there exists some constant C_3 such that (2.1) holds for any cube Q with $\ell(Q) \leq 1$ and C_3 instead of C_2, (2.2) holds for any two (ρ,β_ρ)-doubling cubes $Q \subset R$ with $\ell(Q) \leq 1$ and C_3 instead of C_2, and

$$\frac{1}{\mu(\eta Q)}\int_Q |f(x)|\, d\mu(x) \leq C_3 \tag{2.4}$$

for any cube Q with $\ell(Q) > 1$. We set

$$\|f\|_{\mathfrak{rbmo}(\mu)} = \inf\{C_3 : (2.1), (2.2) \text{ and } (2.4) \text{ hold}\}. \tag{2.5}$$

We should have referred to the choice of constants η, ρ and β in the terminology, but it is proved in [7] and [9] that $RBMO(\mu)$ and $\mathfrak{rbmo}(\mu)$ are independent of their choice. We also have (see Proposition 2.5 of [7] and Proposition 2.2 of [9]) that $(RBMO(\mu), \|\cdot\|_{RBMO(\mu)})$ and $(\mathfrak{rbmo}(\mu), \|\cdot\|_{\mathfrak{rbmo}(\mu)})$ are Banach spaces of functions (modulo additive constants).

We have that $S_{Q,R} \approx 1 + \delta(Q,R)$ (see [8]), where

$$\delta(Q,R) = \max\left(\int_{Q_R\setminus Q}\frac{d\mu(x)}{|x-x_Q|^n}, \int_{R_Q\setminus R}\frac{d\mu(x)}{|x-x_R|^n}\right), \tag{2.6}$$

and there exits a constant $\kappa > 0$ such that for all cubes $Q \subset R$ we have

$$\delta(Q,R) \leq \kappa \left(1 + \log(\frac{\ell(R)}{\ell(Q)})\right). \tag{2.7}$$

Lemma 2.3. *Let $f \in RBMO(\mu)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then the pointwise product $f\varphi \in RBMO(\mu)$. Moreover, if $f \in \mathfrak{rbmo}(\mu)$ then $f\varphi \in \mathfrak{rbmo}(\mu)$.*

Proof. Let $f \in RBMO(\mu)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with support in the cube Q_0. We assume without loss of generality that $f_{2\tilde{Q}_0} = 0$. The point wise product $f\varphi$ belongs to $RBMO(\mu)$ if and only if for some real number $\rho > 1$, there exists $C > 0$ and a collection of numbers $\{C_Q(f\varphi)\}_Q$ (i.e for each cube Q, there exists $C_Q(f\varphi) \in \mathbb{R}$) such that

$$\int_Q |(f\varphi)(x) - C_Q(f\varphi)|\, d\mu(x) \leq C \tag{2.8}$$

and

$$|C_Q(f\varphi) - C_R(f\varphi)| \leq CS_{Q,R} \text{ for any two cubes } Q \subset R. \tag{2.9}$$

A-*The choice of the numbers $C_Q(f\varphi)$ satisfying (2.8)*

Let Q be a cube in $\mathbb{R}^d$. If

1. $\mu(Q \cap Q_0) = 0$, or
2. $\mu(Q \cap Q_0) > 0$ and $Q \not\subset 2Q_0$

then we take $C_Q(f\varphi) = 0$. In the case (1) we have $\int_Q |f\varphi|\, d\mu = 0$ while in the case (2) we have $Q_0 \subset 5Q$ so that

$$\int_Q |f\varphi|\, d\mu = \int_{Q\cap Q_0} |f\varphi|\, d\mu \leq \int_{Q_0} |f\varphi|\, d\mu \leq C\, \|\varphi\|_{L^\infty} \|f\|_{RBMO(\mu)}\, \mu(\rho Q).$$

for any $\rho > 5$. We suppose now that $\mu(Q \cap Q_0) > 0$ and $Q \subset 2Q_0$.

We put $C_Q(f\varphi) = f_{\tilde{Q}}\varphi_Q$. It follows that

$$\begin{aligned} \int_Q \left|f\varphi - f_{\tilde{Q}}\varphi_Q\right| d\mu &= \int_Q \left|(f - f_{\tilde{Q}})\varphi + f_{\tilde{Q}}(\varphi - \varphi_Q)\right| d\mu \\ &\leq \|\varphi\|_{L^\infty} \|f\|_{RBMO(\mu)}\, \mu(\rho Q) + \left|f_{\tilde{Q}}\right| \int_Q |\varphi - \varphi_Q|\, d\mu. \end{aligned}$$

But

$$\begin{aligned} \left|f_{\tilde{Q}}\right| &= \left|f_{\tilde{Q}} - f_{2\tilde{Q}_0}\right| \leq S_{Q,2Q_0} \|f\|_{RBMO(\mu)} \\ &\leq C(1 + \delta_{(Q,2Q_0)}) \|f\|_{RBMO(\mu)} \leq C(1 + \log(\frac{2\ell(Q_0)}{\ell(Q)})) \|f\|_{RBMO(\mu)}, \end{aligned}$$

according to Lemma 2.4 of [8]. So that taking into consideration the following classical result

$$\int_Q |\varphi - \varphi_Q|\, d\mu \leq C\, \|\nabla\varphi\|_{L^\infty} \ell(Q)\mu(Q)$$

and the fact that $2\ell(Q_0)\geq \ell(Q)$, we obtain

$$\begin{aligned} |f_{\tilde{Q}}|\int_Q |(\varphi-\varphi_Q)|\,d\mu &\leq C(1+\log(\frac{2\ell(Q_0)}{\ell(Q)}))\ell(Q)\mu(Q)\,\|f\|_{RBMO(\mu)} \\ &\leq C\mu(Q)\,\|f\|_{RBMO(\mu)}. \end{aligned}$$

B-*Prove that the collection satisfy (2.9)*

Let $Q\subset R$ be two cubes. If $R\cap Q_0=\emptyset$ or $Q\not\subset 2Q_0$, then $C_Q(f\varphi)=C_R(f\varphi)=0$. Thus there is nothing to prove.

We suppose that $R\cap Q_0\neq\emptyset$ and $Q\subset 2Q_0$.

If $R\not\subset 2Q_0$ then $C_R(f\varphi)=0$ and $Q_0\subset 5R$, so that

$$\begin{aligned} \left|f_{\tilde{Q}}\varphi_Q\right| &\leq \|\varphi\|_{L^\infty}\left|f_{\tilde{Q}}-f_{2\tilde{Q}_0}\right|\leq \|\varphi\|_{L^\infty}S_{Q,2Q_0}\,\|f\|_{RBMO(\mu)} \\ &\leq \|\varphi\|_{L^\infty}S_{Q,10R}\,\|f\|_{RBMO(\mu)}\leq C\,\|\varphi\|_{L^\infty}S_{Q,R}\,\|f\|_{RBMO(\mu)}. \end{aligned}$$

If $R\subset 2Q_0$, then

$$\begin{aligned} |C_R(f\varphi)-C_Q(f\varphi)| &= \left|f_{\tilde{R}}\varphi_R-f_{\tilde{Q}}\varphi_Q\right|\leq \|\varphi\|_{L^\infty}\left|f_{\tilde{R}}-f_{\tilde{Q}}\right|+|f_{\tilde{R}}|\,|\varphi_R-\varphi_Q| \\ &\leq \|\varphi\|_{L^\infty}S_{Q,R}\,\|f\|_{RBMO(\mu)}+|f_{\tilde{R}}|\,|\varphi_R-\varphi_Q|. \end{aligned}$$

Let us estimate the second term.

$$\begin{aligned} |f_{\tilde{R}}|\,|\varphi_R-\varphi_Q| &\leq C\,|f_{\tilde{R}}|\,(\ell(R)+\ell(Q)+\mathrm{dist}(x_Q,x_R)) \\ &\leq C(1+|f_{\tilde{R}}|\,\mathrm{dist}(x_Q,x_R)), \end{aligned}$$

where x_Q and x_R denote the centers of the cubes Q and R respectively. But $\mathrm{dist}(x_Q,x_R)\leq C\ell(Q_R)$ and $|f_{\tilde{R}}|\leq\left|f_{\tilde{Q}_R}\right|+\left|f_{\tilde{Q}_R}-f_{\tilde{R}}\right|$, which leads to

$$\begin{aligned} |f_{\tilde{R}}|\,\mathrm{dist}(Q,R) &\leq C\ell(Q_R)\left(\left|f_{\tilde{Q}_R}\right|+\left|f_{\tilde{Q}_R}-f_{\tilde{R}}\right|\right) \\ &\leq C\,\|f\|_{RBMO(\mu)}+S_{R,Q_R}\,\|f\|_{RBMO(\mu)}\leq C\,\|f\|_{RBMO(\mu)}. \end{aligned}$$

The result follow.

Let us consider know the particular case where f belongs to $\mathfrak{rbmo}(\mu)$. For any cube Q such that $\ell(Q)>1$, we have

$$|C_Q(f\varphi)|\leq\left|f_{\tilde{Q}}\right||\varphi_Q|\leq\|\varphi\|_{L^\infty}\,\|f\|_{\mathfrak{rbmo}(\mu)}\,\mu(\eta\tilde{Q})/\mu(\tilde{Q})\leq C\,\|\varphi\|_{L^\infty(\mu)}\,\|f\|_{\mathfrak{rbmo}(\mu)}$$

for some positive constant C and fixed $1<\eta\leq\rho$, since $\ell(\tilde{Q})\geq\ell(Q)$. It follows that $f\varphi\in\mathfrak{rbmo}(\mu)$. $\square$

Inequalities of John-Nirenberg type are valid in both spaces. More precisely we have

Theorem 2.4. *[7] Let $f\in RBMO(\mu)$. For any cube Q and any $\lambda>0$, we have*

$$\mu\left(\left\{x\in Q:\left|f(x)-f_{\tilde{Q}}\right|>\lambda\right\}\right)\leq C_4\mu(\rho Q)\exp\left(-\frac{C_5\lambda}{\|f\|_{RBMO(\mu)}}\right), \tag{2.10}$$

where the constants $C_4>0$ and $C_5>0$ depend only on $\rho>1$

As we can see in Theorem 2.6 of [9], one can replace in the previous theorem the space $RBMO(\mu)$ by its local version $\mathfrak{rbmo}(\mu)$ provided the cube Q satisfies $\ell(Q) \leq 1$, while for cubes Q such that $\ell(Q) > 1$ we have $\mu(\{x \in Q : |f(x)| > \lambda\}) \leq C_4 \mu(\rho Q) \exp\left(-\frac{C_5 \lambda}{\|f\|_{\mathfrak{rbmo}(\mu)}}\right)$. An immediate consequence of this result is that there exists a non-negative constant C_6, which can be chosen as big as we like, such that for all cube Q and const $\not\equiv f \in RBMO(\mu)$,

$$\frac{1}{\mu(\rho Q)} \int_Q \exp\left(\frac{\left|f - f_{\tilde{Q}}\right|}{C_6 \|f\|_{RBMO(\mu)}}\right) d\mu \leq 1. \tag{2.11}$$

We also have the following:

Lemma 2.5. *Let const $\not\equiv f \in RBMO(\mu)$ and $\mathbb{Q}$ the unit cube. We have*

$$\int_{\mathbb{R}^d} \frac{\left(\exp\left(\frac{\left|f(x) - f_{\tilde{\mathbb{Q}}}\right|}{k}\right) - 1\right) d\mu(x)}{(1+|x|)^{2n+\kappa}} \leq 1 \tag{2.12}$$

where $k = C_7 \|f\|_{RBMO(\mu)}$.

Proof. Let $f \in RBMO(\mu)$ with $\|f\|_{RBMO(\mu)} \neq 0$. We have

$$\int_{\mathbb{R}^d} \frac{e^{\frac{\left|f(x)-f_{\tilde{\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}} - 1}{(1+|x|)^{2n+\kappa}} d\mu(x) = \int_{\mathbb{Q}} \frac{e^{\frac{\left|f(x)-f_{\tilde{\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}} - 1}{(1+|x|)^{2n+\kappa}} d\mu(x) + \int_{\mathbb{Q}^c} \frac{e^{\frac{\left|f(x)-f_{\tilde{\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}} - 1}{(1+|x|)^{2n+\kappa}} d\mu(x),$$

where $\mathbb{Q}^c = \mathbb{R}^d \setminus \mathbb{Q}$. The first term in the right hand side is less that $\mu(\rho\mathbb{Q})$. For the second term, we have

$$\begin{aligned}
\int_{\mathbb{Q}^c} \frac{e^{\frac{\left|f(x)-f_{\tilde{\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}} - 1}{(1+|x|)^{2n+\kappa}} d\mu(x) &= \sum_{k=0}^{\infty} \int_{2^{k+1}\mathbb{Q} \setminus 2^k \mathbb{Q}} \frac{e^{\frac{\left|f(x)-f_{\tilde{\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}} - 1}{(1+|x|)^{2n+\kappa}} d\mu(x) \\
&\leq C \sum_{k=0}^{\infty} 2^{-(2n+\kappa)k} \int_{2^{k+1}\mathbb{Q}} \left(e^{\frac{\left|f(x)-f_{\tilde{\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}} - 1\right) d\mu(x).
\end{aligned}$$

Furthermore, there exists a non-negative constant K such that

$$\left|f_{\tilde{R}} - f_{\tilde{Q}}\right| \leq K S_{Q,R} \|f\|_{RBMO(\mu)} \text{ for two cubes } Q \subset R, \tag{2.13}$$

as we can see in the proof of Lemma 2.8 in [7]. We also have $S_{\mathbb{Q},2^{k+1}\mathbb{Q}} \leq (k+2)$, which leads to $\left|f_{\tilde{\mathbb{Q}}} - f_{\widetilde{2^{k+1}\mathbb{Q}}}\right| \leq \log(2^{\frac{K}{\log 2}(k+2)}) \|f\|_{RBMO(\mu)}$.

Hence

$$\begin{aligned}
&\sum_{k=0}^{\infty} 2^{-(2n+\kappa)k} \int_{2^{k+1}\mathbb{Q}} \left(e^{\frac{\left|f(x)-f_{\tilde{\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}} - 1\right) d\mu(x) \\
&\leq \sum_{k=0}^{\infty} 2^{-(2n+\kappa)k} 2^{\frac{K}{C_6 \log 2}(k+2)} \int_{2^{k+1}\mathbb{Q}} \left(e^{\frac{\left|f(x)-f_{\widetilde{2^{k+1}\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}} - 1\right) d\mu(x) \\
&\leq C \sum_{k=0}^{\infty} 2^{(-n-\kappa+\frac{K}{C_6 \log 2})k}.
\end{aligned}$$

If we choose $C_6 > \frac{K}{(n+\kappa)\log 2}$ then the above series converges. Finally we have

$$\int_{\mathbb{R}^d} \frac{e^{\frac{\left|f(x)-f_{\tilde{\mathbb{Q}}}\right|}{C_6\|f\|_{RBMO(\mu)}}}-1}{(1+|x|)^{2n+\kappa}} d\mu(x) \leq K_1, \tag{2.14}$$

where K_1 is a non-negative constant not depending on f.

Thus the result follows from taking $C_7 = \max(C_6, K_1 C_6)$. $\square$

3 Some properties of Orlicz and Hardy-Orlicz space

For the definition of Hardy-Orlicz space, we need the maximal characterization of $\mathcal{H}^1(\mu)$ given in [8].

Let $f \in L^1_{loc}(\mu)$, we set

$$\mathcal{M} f(x) = \sup_{\varphi \in F(x)} \left| \int_{\mathbb{R}^d} f\varphi d\mu \right|, \tag{3.1}$$

where for $x \in \mathbb{R}^d$, $F(x)$ is the set of $\varphi \in L^1(\mu) \cap \mathcal{C}^1(\mathbb{R}^d)$ satisfying the following conditions:

$$\|\varphi\|_{L^1(\mu)} \leq 1, \tag{3.2}$$

$$0 \leq \varphi(y) \leq \frac{1}{|y-x|^n} \text{ for all } y \in \mathbb{R}^d \tag{3.3}$$

and

$$|\nabla\varphi(y)| \leq \frac{1}{|y-x|^{n+1}} \text{ for all } y \in \mathbb{R}^d. \tag{3.4}$$

Tolsa proved in Theorem 1.2 of [8] that a function $f \in L^1(\mu)$ belongs to the Hardy space $\mathcal{H}^1(\mu)$ if and only if $\int_{\mathbb{R}^d} f d\mu = 0$ and $\mathcal{M} f \in L^1(\mu)$. Moreover, in this case we have

$$\|f\|_{\mathcal{H}^1(\mu)} \approx \|f\|_{L^1(\mu)} + \left\|\mathcal{M} f\right\|_{L^1(\mu)}. \tag{3.5}$$

Hardy-Orlicz spaces are defined via this maximal characterization. We recall that for a continuous function $\mathcal{P} : [0,\infty) \to [0,\infty)$ increasing from zero to infinity (but not necessarily convex), the Orlicz space $L^{\mathcal{P}}(\mu)$ consists of μ-measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^{\mathcal{P}}(\mu)} := \inf\left\{k > 0 : \int_{\mathbb{R}^d} \mathcal{P}\left(k^{-1}|f|\right) d\mu \leq 1\right\} < \infty. \tag{3.6}$$

In general, the nonlinear functional $\|\cdot\|_{L^{\mathcal{P}}(\mu)}$ need not satisfy the triangle inequality. It is well known that $L^{\mathcal{P}}(\mu)$ is a complete linear metric space, see [12]. The $L^{\mathcal{P}}$-distance between f and g is given by

$$\mathrm{dist}_{\mathcal{P}}[f,g] := \inf\left\{\rho > 0 : \int_{\mathbb{R}^d} \mathcal{P}\left(\rho^{-1}|f-g|\right) d\mu \leq \rho\right\} < \infty. \tag{3.7}$$

The Hardy-Orlicz space $\mathcal{H}^{\mathcal{P}}(\mu)$ consists of local integrable function f such that $\mathcal{M} f \in L^{\mathcal{P}}(\mu)$. We put

$$\|f\|_{\mathcal{H}^{\mathcal{P}}(\mu)} = \left\|\mathcal{M} f\right\|_{L^{\mathcal{P}}(\mu)}. \tag{3.8}$$

It comes from what precede that $\mathcal{H}^{\mathcal{P}}(\mu)$ is a complete linear metric space, a Banach space when $\mathcal{P}$ is convex. These spaces have previously been dealt with by many authors, see [13, 14, 15] and further references given there. When we consider the Orlicz function $\wp(t) = \dfrac{t}{\log(e+t)}$, we have the following results given in [1].

• If $\mathrm{dist}_{\wp}[f,g] \leq 1$ then $\|f-g\|_{L^{\wp}(\mu)} \leq \mathrm{dist}_{\wp}[f,g] \leq 1$,

• The sequence $(f_j)_{j>0}$ converges to f in $L^{\wp}(\mu)$ if and only if $\left\|f_j - f\right\|_{L^{\wp}(\mu)} \to 0$,

• We have duality between the Orlicz space $L^{\Xi}(\mu)$ associated to the Orlicz function $\Xi(t) = e^t - 1$ and $L^{\tilde{\wp}}(\mu)$ with $\tilde{\wp}(x) = x\log(e+x)$ in the sense that for $f \in L^{\Xi}(\mu)$ and $g \in L^{\tilde{\wp}}(\mu)$ we have

$$\|fg\|_{L^1(\mu)} \leq \|f\|_{L^{\Xi}(\mu)} \|g\|_{L^{\tilde{\wp}}(\mu)}. \tag{3.9}$$

• For $f,g \in L^{\wp}(\mu)$,we have the following substitute of the additivity

$$\|f+g\|_{L^{\wp}(\mu)} \leq 4\|f\|_{L^{\wp}(\mu)} + 4\|g\|_{L^{\wp}(\mu)}. \tag{3.10}$$

• Let

$$d\sigma = \frac{d\mu}{(1+|x|)^{2n+\kappa}} \text{ and } d\nu = \frac{d\mu}{\log(e+|x|)}, \tag{3.11}$$

for $f \in L^{\Xi}(\sigma)$ and $g \in L^1(\mu)$, we have $fg \in L^{\wp}(\nu)$ and

$$\|fg\|_{L^{\wp}(\nu)} \leq C\|f\|_{L^{\Xi}(\sigma)} \|g\|_{L^1(\mu)}. \tag{3.12}$$

and for $f \in RBMO(\mu)$ and $g \in L^1(\mu)$,

$$\|fg\|_{L^{\wp}(\nu)} \leq C\|f\|_{RBMO(\mu)^+} \|g\|_{L^1(\mu)}, \tag{3.13}$$

where $\|f\|_{RBMO^+(\mu)} = \|f\|_{RBMO(\mu)} + \left|f_{\tilde{\mathbb{Q}}}\right|$

4 Proof of the main results

Proof of Theorem 1.2. Let $f \in RBMO(\mu)$ and $h \in \mathcal{H}^1(\mu)$, h having the p-atomic blocks decomposition given in (1.8), i.e.

$$h = \sum_j b_j, \tag{4.1}$$

where $b_j = \sum_{i=1}^{\infty} \lambda_{ij} a_{ij}$ is the atomic-block supported in the cube R_j, a_{ij} supported in the cube $Q_{ij} \subset R_j$ and $\left\|a_{ij}\right\|_{L^\infty(\mu)} \leq \mu(\rho Q_{ij})^{-1}\left(S_{Q_{ij},R_j}\right)^{-1}$.

We have

$$\begin{aligned}
\left\|\lambda_{ij}\left(f - f_{\tilde{R}_j}\right)a_{ij}\right\|_{L^1(\mu)} &\leq \left|\lambda_{ij}\right| \int_{Q_{ij}} \left|f - f_{\tilde{R}_j}\right| \left|a_{ij}\right| d\mu \\
&\leq \left|\lambda_{ij}\right| \left(\int_{Q_{ij}} \left|f - f_{\tilde{Q}_{ij}}\right| \left|a_{ij}\right| d\mu + \int_{Q_{ij}} \left|f_{\tilde{R}_j} - f_{\tilde{Q}_{ij}}\right| \left|a_{ij}\right| d\mu\right) \\
&\leq C\left|\lambda_{ij}\right| \|f\|_{RBMO(\mu)},
\end{aligned}$$

according to Inequalities (2.13) and (1.6), which proves that the first series $\sum_{j=1}^{\infty}\left(f - f_{\tilde{R}_j}\right) b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_{ij}\left(f - f_{\tilde{R}_j}\right) a_{ij}$ converges normally in $L^1(\mu)$, since the atomic decomposition theorem asserts that the double series $\sum_{i,j}\left|\lambda_{ij}\right|$ converges. It remains to prove the convergence of

$$S = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\lambda_{ij} f_{\tilde{R}_j} a_{ij}\right) = \sum_{j=1}^{\infty} f_{\tilde{R}_j} b_j \tag{4.2}$$

in $\mathcal{H}^{\wp}(\nu)$. For this purpose, we have to prove that the sequence $S_N = \mathcal{M}\left(\sum_{j=1}^{N} f_{\tilde{R}_j} b_j\right)$ is Cauchy in $L^{\wp}(\nu)$. This is equivalent to prove that $\lim\limits_{l\to\infty}\left\|\mathcal{M}\left(\tilde{S}_l^k\right)\right\|_{L^{\wp}(\nu)} = 0$, where

$$\tilde{S}_l^k = \sum_{j=l}^{k} f_{\tilde{R}_j} b_j \text{ with } l \leq k. \tag{4.3}$$

Since

$$\mathcal{M}\left(f_{\tilde{R}_j} b_j\right) \leq \left|f - f_{\tilde{R}_j}\right| \mathcal{M}(b_j) + |f|\,\mathcal{M}(b_j), \tag{4.4}$$

we have that

$$\left\|\mathcal{M}\left(\tilde{S}_l^k\right)\right\|_{L^{\wp}(\nu)} \leq 4\left\|\sum_{j=l}^{k}\left|f - f_{\tilde{R}_j}\right|\mathcal{M}(b_j)\right\|_{L^1(\mu)} + 4\left\|\sum_{j=l}^{k}|f|\,\mathcal{M}(b_j)\right\|_{L^{\wp}(\nu)}, \tag{4.5}$$

according to (3.10) and the fact that $\|f\|_{L^{\wp}(\mu)} \leq \|f\|_{L^1(\mu)}$ for all measurable functions f. Let us consider the first term in the second member of (4.5). We have

$$\left\|\sum_{j=l}^{k}\left|f - f_{\tilde{R}_j}\right|\mathcal{M}(b_j)\right\|_{L^1(\mu)} \leq \sum_{j=l}^{k}\left\|\sum_{i=1}^{\infty}|\lambda_{ij}|\left(\left|f - f_{\tilde{Q}_{ij}}\right| + \left|f_{\tilde{R}_j} - f_{\tilde{Q}_{ij}}\right|\right)\mathcal{M}(a_{ij})\right\|_{L^1(\mu)}, \tag{4.6}$$

since $\mathcal{M}(b_j) \leq \sum_{i=1}^{\infty}\left|\lambda_{ij}\right|\mathcal{M}(a_{ij})$. From the definition of $\mathcal{M}(a_{ij})$, we have

$$\mathcal{M}(a_{ij})(x) \leq \mu(\rho Q_{ij})^{-1}\left(S_{Q_{ij},R_j}\right)^{-1}, \tag{4.7}$$

so that taking into consideration relation (2.13), we obtain

$$\left\|\left(\left|f - f_{\tilde{Q}_{ij}}\right| + \left|f_{\tilde{R}_j} - f_{\tilde{Q}_{ij}}\right|\right)\mathcal{M}(a_{ij})\right\|_{L^1(\mu)} \leq C\,\|f\|_{RBMO(\mu)}. \tag{4.8}$$

Thus

$$\lim_{l\to\infty}\left\|\sum_{j=l}^{k}\left|f - f_{\tilde{R}_j}\right|\mathcal{M}(b_j)\right\|_{L^1(\mu)} = 0, \tag{4.9}$$

since the double series $\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\left|\lambda_{ij}\right|\right)$ converges. Let us consider now the series

$$\left\|\sum_{j=l}^{k}|f|\,\mathcal{M}(b_j)\right\|_{L^{\wp}(\nu)} = \left\||f|\sum_{j=l}^{k}\mathcal{M}(b_j)\right\|_{L^{\wp}(\nu)}.$$

We have

$$\left\| \sum_{j=l}^{k} \mathcal{M}\,(b_j) \right\|_{L^1(\mu)} \leq C \sum_{j=l}^{k} \left(\sum_{i=1}^{\infty} |\lambda_{ij}| \right), \tag{4.10}$$

according to Lemma 3.1 of [8]. Furthermore, we have

$$\left\| |f| \sum_{j=l}^{k} \mathcal{M}\,(b_j) \right\|_{L^{\wp}(\nu)} \leq \|f\|_{RBMO^+(\mu)} \left\| \sum_{j=l}^{k} \mathcal{M}\,(b_j) \right\|_{L^1(\mu)}, \tag{4.11}$$

according to (3.13). □

Definition 4.1. *([2]) $L_*^{\wp}$ is the space of functions f such that*

$$\|f\|_{L_*^{\wp}} := \sum_{j \in \mathbb{Z}^n} \|f\|_{L^{\wp}(j+\mathbb{Q})} < \infty,$$

where $\mathbb{Q}$ is the unit cube centered at 0.

We accordingly define $\mathcal{H}_*^{\wp}$. Using the concavity described above, we have $\wp(st) \leq Cs\wp(t)$ for $s > 1$. It follows that $L^{\wp}$ is contained in $L_*^{\wp}$ as a consequence of the fact that $\|f\|_{L^{\wp}(j+\mathbb{Q})} \leq \int_{j+\mathbb{Q}} \wp(|f|)d\mu(x)$. The converse inclusion is not true.

Theorem 4.2. *For $h \in \mathcal{H}^1(\mu)$ and $f \in \mathfrak{rbmo}(\mu)$, the product $f \times h$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion*

$$f \times h \in L^1(\mu) + \mathcal{H}_*^{\wp}(\mu). \tag{4.12}$$

Proof. The proof is inspired by the one given in [2] in the case of Lebesgue measure. Let $f \in \mathfrak{rbmo}(\mu)$ and $h \in \mathcal{H}^1(\mu)$ being as in the proof of Theorem 1.2. The series

$$\sum_j \left(\sum_i \lambda_{ij} (f - f_{\tilde{R}_j}) a_{ij} \right), \ \sum_j (f - f_{\tilde{R}_j}) \mathcal{M}\,(b_j) \text{ and } \sum_j \mathcal{M}\,(b_j) \tag{4.13}$$

converge normally in $L^1(\mu)$ and

$$\mathcal{M}\,\left(\sum_j b_j f_{\tilde{R}_j} \right) \leq \sum_j \left| f - f_{\tilde{R}_j} \right| \mathcal{M}\,(b_j) + |f| \sum_j \mathcal{M}\,(b_j). \tag{4.14}$$

Thus we just have to prove that the second term in the right hand side of (4.14) is in $L_*^{\wp}(\mu)$. Let Q be a cube of side length 1. By John-Nirenberg inequality on $\mathfrak{rbmo}(\mu)$, we have that there exists $c_7 > 0$ (we can choose any number greater than $\frac{1}{c_5} + \frac{c_4 2^n}{c_5}$) such that

$$\int_Q \left(e^{\frac{|f(x)|}{c_7 \|f(x)\|_{\mathfrak{rbmo}(\mu)}}} - 1 \right) d\mu(x) \leq 1. \tag{4.15}$$

We claim that for $\psi \in L^1(\mu)$

$$\|f\psi\|_{L^{\wp}(Q)} \leq C \|f\|_{\mathfrak{rbmo}(\mu)} \int_Q |\psi|\, d\mu. \tag{4.16}$$

In fact, by homogeneity, we can assume that $c_7\|f\|_{\mathfrak{rbmo}(\mu)}=1$ and it is sufficient to find some constant c such that for $\int_Q|\psi|\,d\mu=c$ we have

$$\int_Q\frac{|f\psi|}{\log(e+|f\psi|)}d\mu\le 1.$$

We have

$$\int_Q\frac{|f\psi|}{\log(e+|f\psi|)}d\mu=\int_{Q\cap\{|f|\le 1\}}\frac{|f\psi|}{\log(e+|f\psi|)}d\mu+\int_{Q\cap\{|f|>1\}}\frac{|f\psi|}{\log(e+|f\psi|)}d\mu. \quad (4.17)$$

The first term in the second member is bounded by $\int_Q|\psi|\,d\mu$ and for the second term, we have

$$\begin{aligned}\int_{Q\cap\{|f|>1\}}\frac{|f\psi|}{\log(e+|f\psi|)}d\mu &\le \int_{Q\cap\{|f|>1\}}|f|\frac{|\psi|}{\log(e+|\psi|)}d\mu\\ &\le \|f\|_{L^{\Xi}(Q)}\left\|\frac{|\psi|}{\log(e+|\psi|)}\right\|_{L^{\tilde{\wp}}(Q)}\le C\left\|\frac{|\psi|}{\log(e+|\psi|)}\right\|_{L^{\tilde{\wp}}(Q)}.\end{aligned}$$

But

$$\begin{aligned}\int_Q\frac{|\psi|}{\log(e+|\psi|)}\log\left(e+\frac{|\psi|}{\log(e+|\psi|)}\right)d\mu &\le \int_Q\frac{|\psi|}{\log(e+|\psi|)}\log(e+|\psi|)\,d\mu\\ &\le \int_Q|\psi|\,d\mu\end{aligned}$$

Thus if $c<\frac{1}{2}$ and $\int_Q|\psi|\,d\mu=c$ the result follows. We have an estimate for each cube $j+\mathbb{Q}$, and sum up. This finishes the proof. $\square$

Since we do not have any maximal function characterization of the local Hardy spaces on non-homogeneous space in the literature, we are going to define the local space corresponding to $\mathcal{H}^1_*$ in the same manner as in [2]. For this purpose, we put

$$\mathcal{M}^{(1)}f(x)=\sup_{F_{loc}(x)}\left|\int f\varphi d\mu\right|, \quad (4.18)$$

where $F_{loc}(x)$ denotes the set of elements belonging to $F(x)$ as define in Section 3, but having their support in the cube $Q(x,1)$ centered at x with side length 1. A locally integrable function f belongs to the space $\mathfrak{h}^{\wp}_*(\mu)$ if $\mathcal{M}^{(1)}f\in L^{\wp}_*(\mu)$.

Proposition 4.3. *For h a function in $\mathfrak{h}^1(\mu)$ and b a function in $\mathfrak{rbmo}(\mu)$, the product $b\times h$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion*

$$b\times h\in L^1(\mu)+\mathfrak{h}^{\wp}_*(\mu). \quad (4.19)$$

Proof. Let $f\in\mathfrak{rbmo}(\mu)$ and $h\in\mathfrak{h}^1(\mu)$ with $h=\sum_j b_j$ where b_j's are atomic blocks or blocks. Since we do not use the cancellation property of b_j's to prove that the $\sum_j(f-f_{\tilde{R}_j})b_j$ converge absolutely in $L^1(\mu)$, it follows that the result remains true in this case. Thus we just

have to prove that the second term belongs to the amalgam space $\mathfrak{h}_*^{\wp}(\mu)$. This immediate if we prove that for any bock b_j, the quantity $\left\|\mathcal{M}^{(1)}b_j\right\|_{L^1(\mu)}$ is bounded by a constant which is independent on b_j. Let $b_j = \sum_{i=1}^{\infty} \lambda_{ij} a_{ij}$, where a_{ij} is supported in the cube $Q_{ij} \subset R_j$ and satisfy $\left\|a_{ij}\right\|_{L^\infty(\mu)} \leq \left(\mu(2Q_{ij})S_{Q_{ij},R_j}\right)^{-1}$. For every integer i, we have

$$\mathcal{M}^{(1)}a_{ij}(x) \leq \left(\mu(2Q_{ij})S_{Q_{ij},R_j}\right)^{-1} \chi_{2R_j}(x), \tag{4.20}$$

where χ_{2R_j} denote the characteristic function of $2R_j$. In fact, if $\varphi \in F_{loc}(x)$ then $\int a_{ij}\varphi d\mu \neq 0$ only if $x \in 2R_j$, since $\ell(R_j) > 1$. Proceeding as in the prove of Proposition 2.6 in [8], we have

$$\int_{\mathbb{R}^d} \mathcal{M}^{(1)}a_{ij}(x)d\mu(x) = \int_{2R_j} \mathcal{M}^{(1)}a_{ij}(x)d\mu(x) \leq C, \tag{4.21}$$

where C is independent of i and j. Then we conclude as in the proof of Theorem 4.2. $\square$

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